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FUZZY NORMAL CONGRUENCES OF INVERSE SEMIGROUPS.

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Abstract. In this paper we introduce the notions of fuzzy kernel, fuzzy trace of a fuzzy congruence on an inverse semigroup, give some of their properties in connection with fuzzy normal congruences. As a continuation of these works, we define a fuzzy normal congruence as a fuzzy relation and characterize fuzzy normal congruences in the framework of fuzzy relations.

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1. Introduction

The study of fuzzy relations began with Sanchez[7], Later Nemitz[4] and Murali[3] studied the concept of fuzzy equivalence relations. The compatibility of fuzzy equivalence relations with underlying algebraic structures was one of the ways to introduce fuzzy quotient structures, this was done by Samhan[6] when he introduced fuzzy congruences on semigroups. C.H. Seo, K.H. Han, Y.O. Sung, H.C. Eun in [8] characterize fuzzy difunctional relations and prove some properties in connetion with fuzzy difunctional relations. Sung[9] obtain several results which are analogs of some basic theorems of group theory.

In the present note, we carry on the investigation of fuzzy normal congruences on inverse semigroup S, by introducing the notions of fuzzy kerneal, fuzzy trace of a fuzzy congruence and characterize fuzzy normal congruences in the framework of fuzzy relations. Throughout I denotes the closed unit interval [0,1], S an inverse semigroup and E_s the set of all idempotents in S. All fuzzy relations are map $R : X \times X \to X$. For $x, y \in I$, $x \vee y = max\,\{x, y\}$ and $x \times y = min\,\{x, y\}$. If R and S are two fuzzy relations on a set X, then $R \subseteq S$ means that $R(x, y) \leq S(x, y)$ for all $x, y \in X$.

2. Preliminaries

In this section we review some definitqions and results that will be needed in the sequel. For details we refer to [1, 4, 5].

Definition 2.1. *Let G be a group. A fuzzy subset f of G is called a fuzzy subgroup of G if.*

(1) $f(xy) \geq f(x) \wedge f(y)$ *for all* x, y *of* G
(2) $f(x^{-1}) = f(x)$ *for all* x *of* G.

Definition 2.2. *Let G be a group. A fuzzy subgroup f of G is called a fuzzy nomal subgroup of G if f(xy)=f(yx) for all x, y of G.*

Definition 2.3. *Let R and S be two fuzzy relations on a set X. Then the product RS is defined by $RS(xy) = \bigvee_{z\in X} (R(x,z) \wedge R(z,y))$ for all $(x,y) \in X \times X$.*

Definition 2.4. *Let X be a nonempty set. and let R be a fuzzy relation on X. Then R is called a fuzzy equivalence relation on X if and only if*

(1) *R is fuzzy relflexive, i.e, $R(x,x) = 1$ for all $x \in X$.*
(2) *R is fuzzy symmetric, i.e, $R^{-1} = R$.*
(3) *R is fuzzy transtive, i.e, $RR \subseteq R$.*

Definition 2.5. *Let X be a semigroup. A fuzzy relation R on X is xalled fuzzy left(right) compatible iff $R(tx,ty) \geq R(x,y)$ for all $x,y,t \in X$ $(R(xt,yt) \geq R(x,y)$ for all $x,y,z \in X)$.*

Definition 2.6. *A fuzzy relation R on a semigroup X is called fuzzy compatible iff $R(ac,bd) \geq R(a,b) \wedge R(c,d)$ for all $a,b,c,d \in X$.*

Definition 2.7. *A Fuzzy compatible equivalence relation on a semigroup X is called a fuzzy congruence on X.*

Definition 2.8. *Let R be a congruence on in a inverse semigroup S. We denote the fuzzy kernel of R by K_R and is defined as $K_R(x) = \bigvee_{e\in E_s} R(x,e)\ \forall x \in S$.*

Definition 2.9. *A fuzzy congruence τ on E_s is called fuzzy normal congruence on E_s if $\tau(s^{-1}es, s^{-1}fs) \geq \tau(e,f)\ \forall e,f \in E_s,\ s \in S$.*

Definition 2.10. *A fuzzy relation R is fuzzy difunctional if it satisfies the condition $RR^{-1}R \subseteq R$, which is equivalence to $RR^{-1}R = R$.*

3. Fuzzy normal congruencces.

In this section we characterize fuzzy normal congruences in the framework of fuzzy relations.

Theorem 3.1. [1]. *If θ is a fuzzy congruence on S, then $\theta(x^{-1},y^{-1}) = \theta(x,y)\ \forall x,y \in S$.*

Theorem 3.2. *Let R be a fuzzy equivalence relation on S. Then R is a fuzzy congruence on S iff R is fuzzy left and right compatible.*

Proof. Assume that R is a fuzzy congruence on S, then we have $R(xz,yz) \geq R(x,y) \wedge R(z,z) = R(x,y)$ for all $x,y,z \in X$, similarly $R(zx,zy) \geq R(x,y)$ can be done. Thus this means that R is fuzzy left and right compatible. Conversely, assume that R is fuzzy left and right compatible, then we have,

$$\begin{aligned} R(xz,yt) &= RR(xz,yt),\ as\ R\ is\ fuzzy\ reflexive\ and\ fuzzy\ transitive \\ &\geq R(xz,yz) \wedge R(yz,yt),\ as\ R\ is\ fuzzy\ transitive \\ &= R(x,y) \wedge R(z,z) \wedge R(y,y) \wedge R(z,t) \\ &= R(z,y) \wedge R(z,t)\ for\ all\ x,y,z,t \in S. \end{aligned}$$

Which yields R is fuzzy compatible. This completes the proof. □

Theorem 3.3. *Let R be a fuzzy congruence on S. Then K_R is a fuzzy subgroup on S.*

Proof. We claim that $K_R(x,y) \geq K_R(x) \wedge K_R(y)$ for all $x, y \in S$. Suppose not, then there exist $x, y \in S$ and $r > 0$ such that $K_R(xy) < r < K_R(x) \wedge K_R(y)$. Considering $K_R(x) > r$ and $K_R(y) > r$, these mean that there exist $e, e' \in S$ such that $R(x, e) > r$ and $R(y, e') > r$. On the other hand, due to $ee' \in S$, we have

$$\begin{aligned} K_R(xy) &= \bigvee_{f \in E_s} R(xy, f) \\ &\geq R(xy, ee') \\ &\geq R(x, e) \wedge R(y, e') \\ &> r. \end{aligned}$$

This complicts to our assumption. Further, for $x \in S$, we have

$$\begin{aligned} K_R(x) &= \bigvee_{e \in E_s} R(x, e) \\ &= \bigvee_{e \in E_s} R(x^{-1}, e^{-1}) \\ &= \bigvee_{e \in E_s} R(x^{-1}, e) \\ &= K_R(x^{-1}). \end{aligned}$$

Therefore K_R is a fuzzy subgroup on S. □

Theorem 3.4. *Let R be a fuzzy congruence on S. Then τ_R is a fuzzy normal congruence on E_s.*

Proof. Since R is a fuzzy congruence on S. We note that τ_R is a fuzzy congruence relation on E_s. To show that τ_R is a fuzzy normal congruence on E_s, it suffices to show that $\tau_R(s^{-1}es, s^{-1}fs) \geq \tau_R(e, f)$ for all $e, f \in E_s$ and $s \in S$. Indeed, $\tau_R(s^{-1}es, s^{-1}fs) = R(s^{-1}es, s^{-1}fs) \geq R(e, f) = \tau_R(e, f)$. This completes the proof. □

Theorem 3.5. *τ is a fuzzy normal congruence on E_s iff τ^{-1} is a fuzzy normal congruence on E_s.*

Proof. Since $\tau = (\tau^{-1})^{-1}$, we note that τ is a fuzzy equivalence relation on E_s iff τ^{-1} is a fuzzy equivalence relation on E_s. Assume that τ is a fuzzy normal congruence on E_s. Then, for $e, f \in E_s$ and $s \in S$, we have $\tau^{-1}(s^{-1}es, s^{-1}fs) = \tau(s^{-1}fs, s^{-1}es) \geq \tau(f, e) = \tau^{-1}(e, f)$, and so τ^{-1} is a fuzzy normal congruence on E_s. Conversely, assume that τ^{-1} is a fuzzy normal congruence on E_s. As seen in above argument, $\tau = (\tau^{-1})^{-1}$ is a fuzzy normal congruence on E_s. This completes the proof. □

Theorem 3.6. *Let f be a fuzzy normal subgroup of a group S with $f(e) = 1$. Then the fuzzy relation R_f defined by $R_f(a, b) = f(a^{-1}b)$ for all $a, b \in S$, is a fuzzy normal congruence on E_s, where e is the identity of G.*

Proof. Let a be a element of E_s, then $R_f(a,a) = f(a^{-1}a) = f(e) = 1$, and so R_f is fuzzy reflexive on E_s. For any elements $a, b \in E_s$, we have $R_f(a,b) = f(a^{-1}b) = f(b^{-1}a) = R_f(a,b)$, and so R_f is fuzzy symmetric. To show that R_f is fuzzy transitive, let a, b, x be any elements of E_s. Then we have,

$$\begin{aligned} R_fR_f(a,b) &= \bigvee_{t\in E_s} (R_f(a,t) \wedge R_f(t,b)) \\ &= \bigvee_{t\in E_s} (f(a^{-1}t) \wedge f(t^{-1}b) \\ &\leq f(a^{-1}b),\ as\ f\ is\ a\ fuzzy\ subgroup \\ &= R_f(a,b). \end{aligned}$$

Which yields R_f is fuzzy transitive on E_s. Thus R_f is a fuzzy equivalence relation on E_s. Let a, b, x be any elements of E_s, then $R_f(xa, xb) = f((xa)^{-1}(xb)) = f(a^{-1}b) = R_f(a,b)$. Similarly $R_f(ax, bx) = R_f(a,b)$ can be done. This implies that R_f is a fuzzy congruence on E_s. Finally, for $e, f \in E_s$ and $s \in S$, we then have

$$\begin{aligned} R_f(s^{-1}es, s^{-1}fs) &= f((s^{-1}es)^{-1}(s^{-1}fs)) \\ &= f(s^{-1}efs) \\ &= f((efs)(s^{-1})),\ as\ f\ is\ a\ fuzzy\ normal\ subgroup \\ &= f(ef) \\ &= f(e^{-1}f) \\ &= R_f(e,f). \end{aligned}$$

Therefore R_f is a fuzzy normal congruence on E_s .

□

Theorem 3.7. [2]. *Let f and g be fuzzy normal subgroups of a group S. Then fg is also a fuzzy normal subgroup of S.*

Theorem 3.8. *Let f and g be fuzzy normal subgroup of a group S. Then R_fg is a fuzzy normal congruence on E_s.*

Proof. It follows from Theorem 3.6 and Theorem 3.7. □

Theorem 3.9. *If R, S are fuzzy normal congruences on E_s such that $RS = SR$, then RS is a fuzzy normal congruence on E_s.*

Proof. Let x be any element of E_s, then $RS(x,x) = \bigvee_{t\in E_s} (R(x,t) \wedge S(t,x)) \geq R(x,x) \wedge S(x,x) = 1$, and so RS is fuzzy reflexive on E_s. And we have $(RS)^{-1} = S^{-1}R^{-1} = SR = RS$, this means that RS is fuzzy symmetric. In addition, $(RS)(RS) = R(SR)S = (RR)(SS) \subseteq RS$, hence RS is fuzzy transitive. Thus RS is a fuzzy equivalence relation on E_s. Now we prove that RS is fuzzy compatible, let $x, y, z \in E_s$ then we prove that $RS(zx, zy) \geq RS(x,y)$. Suppose not then there exist $x, y, z \in S$ and $r > 0$ such that

$RS(zx, zy) < r < RS(x, y)$. $RS(x, y)$ means that there exists $t \in E_s$ such that $R(x, t) > r$ and $S(t, y) > r$. On the other hand,

$$\begin{aligned} RS(zx, zy) &= \bigvee_{s \in E_s} (R(zx, s) \wedge S(s, zy)) \\ &\geq R(zx, zt) \wedge S(zt, zy) \ for\ this\ s = zt \\ &= R(x, t) \wedge S(t, y),\ as\ R\ is\ fuzzy\ left\ and\ right\ compatible \\ &> r. \end{aligned}$$

This contradicts to our assumption. Which show that $RS(zx, zy) \geq RS(x, y)$ for all $x, y, z \in E_s$. Similarly, $(RS)(xz, yz) \geq RS(x, y)$ can be done. Therefore RS is a fuzzy congruence on E_s. Finally we prove that $RS(s^{-1}es, s^{-1}fs) \geq RS(e, f)$ for all $e, f \in E_s$ and $s \in S$. Suppose not, then there exist $e, f \in E_s$, $s \in S$ and $r > 0$ such that $RS(s^{-1}es, s^{-1}fs) < r < RS(e, f)$. But $RS(e, f) > r$ means that there exist $g \in E_s$ such that $R(e, g) > r$ and $S(g, f) > r$. But we have

$$\begin{aligned} RS(s^{-1}es, s^{-1}fs) &= \bigvee_{t \in E_s} (R(s^{-1}es, g) \wedge S(g, s^{-1}fs)) \\ &\geq \bigvee_{t \in E_s} (R(s^{-1}es, s^{-1}gs) \wedge S(s^{-1}gs, s^{-1}fs))\ by\ s^{-1}E_s s \in E_s \\ &\geq R(s^{-1}es, s^{-1}gs) \wedge S(s^{-1}gs, s^{-1}fs) \\ &\geq R(e, g) \wedge S(g, f),\ as\ R,\ S\ are\ fuzzy\ compatible \\ &> r. \end{aligned}$$

This contradicts to our assumption, therefore RS is a fuzzy normal congruence on E_s.

□

Theorem 3.10. *Let S be a group. If R, S are fuzzy normal congruences on E_s, then SR is a fuzzy normal congruence on E_s.*

Proof. In view of Theorem3.9, it suffices to show that $RS = SR$ on E_s. Suppose not, then without loss of generality, we may assume that there exist $x, y \in E_s$ and $r > 0$ such that $RS(x, y) < r < SR(x, y)$. But $SR(x, y) > r$ means that there exists $z \in E_s$ such that $S(x, z) > r$ and $R(z, y) > r$. On the other hand, we have,

$$\begin{aligned} RS(x, y) &= \bigvee_{t \in E_s} (R(x, t) \wedge S(t, y)) \\ &\geq R(x, xz^{-1}y) \wedge S(xz^{-1}y, y)\ for\ t = xz^{-1}y \\ &= R(xz^{-1}z, xz^{-1}y) \wedge S(xz^{-1}y, zz^{-1}y) \\ &\geq R(z, y) \wedge S(x, z) \\ &> r. \end{aligned}$$

This contradicts to our assumption, this entails $RS = SR$. Therefore SR is a fuzzy normal congruence on E_s.

Theorem 3.11. *If R, S are fuzzy normal congruences on E_s, then so is their intersection $R \cap S$.*

Proof. We note that $R \cap S$ is a fuzzy equivalence relation on E_s. We prove that $R \cap S$ is fuzzy compatible. Now for $x, y, z \in E_s$, we have

$$\begin{aligned}(R \cap S)(zx, zy) &= R(zx, zy) \wedge S(zx, zy) \\ &\geq R(x, y) \wedge S(x, y),\ as\ R\ and\ S\ are\ fuzzy\ left\ compatible \\ &= (R \cap S)(e, f).\end{aligned}$$

Similarly, $(R \cap S)(xz, yz) \geq (R \cap S)(x, y)$ can be done. Hence $R \cap S$ is a fuzzy congruence on E_s. Let $e, f \in E_s$ and $s \in S$, then

$$\begin{aligned}(R \cap S)(s^{-1}es, s^{-1}fs) &= R(s^{-1}es, s^{-1}fs) \wedge S(s^{-1}es, s^{-1}fs) \\ &\geq R(e, f) \wedge S(e, f) \\ &= (R \cap S)(e, f).\end{aligned}$$

Thus $R \cap S$ is a fuzzy normal congruence on E_s.

□

Theorem 3.12. *Let R be a fuzzy normal congruence on E_s. For a given $k \in [0, 1]$, a fuzzy relation P defined by $P(x, y) = R(x, y) \vee k$ for all $x, y \in X$, is a fuzzy normal congruence on E_s.*

Proof. For x be any element of E_s, $P(x, x) = R(x, x) \vee k = 1$, and so P is fuzzy reflexive on E_s. $P(x, y) = R(x, x) \vee k = R(y, z) \vee k = P(y, x)$ for all $x, y \in E_s$. This means that P is fuzzy symmetric. Let x, y be any elements of E_s, then we have

$$\begin{aligned}PP(x, y) &= \bigvee_{z \in E_s} (P(x, z) \wedge P(z, y)) \\ &= \bigvee_{z \in E_s} (R(x, z) \vee k) \wedge (R(z, y) \vee k)) \\ &= \bigvee_{z \in E_s} ((R(x, z) \wedge R(z, y)) \vee k) \\ &= RR(x, y) \vee k \\ &= R(x, t) \vee k \\ &= P(x, y).\end{aligned}$$

And so P is fuzzy transitive. Next let a, b, x be any elements of E_s, then

$$\begin{aligned}P(xa, xb) &= R(xa, xb) \vee k \\ &\geq R(a, b) \vee k,\ as\ R\ is\ fuzzy\ left\ compatible \\ &= P(a, b).\end{aligned}$$

Similarly $P(ax, bx) \geq P(a, b)$ can be done. Finally, let $e, f \in E_s$ and $s \in S$, then

$$\begin{aligned} P(s^{-1}es, s^{-1}fs) &= R(s^{-1}es, s^{-1}fs) \vee k \\ &\geq R(e,f) \vee k \\ &= P(e,f). \end{aligned}$$

Therefore P is a fuzzy normal congruence on E_s.

□

Theorem 3.13. *Let R be a fuzzy relation on a group S such that R is fuzzy reflexive and compatible. If R is fuzzy difucntional, then R is a fuzzy normal congruence on E_s.*

Proof. To show that R is a fuzzy equivalence relation on E_s. First we prove that R is fuzzy symmetric;

$$\begin{aligned} R(x,y) &= RR^{-1}R(x,y),\ as\ R\ is\ fuzzy\ difunctional \\ &= R(R^{-1}R)(x,y) \\ &= \bigvee_{z\in E_s} (R(x,z) \wedge \bigvee_{w\in E_s} (R(w,z) \wedge R(w,y))\ for\ z = x \\ &\geq R(x,x) \wedge \bigvee_{w\in E_s} (R(w,x) \wedge R(w,y)) \\ &= \bigvee_{w\in E_s} (R(w,x) \wedge R(w,y)) \\ &\geq R(y,x) \wedge R(y,y)\ for\ w = y \\ &= R(y,x). \end{aligned}$$

This implies that $R(y,x) \geq R(x,y)$ for all $x, y \in E_s$. Similarly, $R(y,x) \geq R(x,y)$ can be done. And so R is fuzzy symmetric. Next we prove that R is fuzzy transitive;

$$\begin{aligned} R(x,y) &= RR^{-1}R(x,y) \\ &= \bigvee_{z\in E_s} R(x,z) \wedge \bigvee_{w\in E_s} (R(w,z) \wedge R(w,y)) \\ &\geq R(x,x) \wedge \bigvee_{w\in E_s} (R(w,x) \wedge R(w,y)\ for\ z = x \\ &= \bigvee_{w\in E_s} (R(w,x) \wedge R(w,y)) \\ &= RR(x,y). \end{aligned}$$

And so $RR \subseteq R$. Thus R is fuzzy transitive. Finally, let $e, f \in E_s$ and $s \in S$, then $R(s^{-1}es, s^{-1}fs) \geq R(es, ef) \geq R(e,f)$. Therefore R is a fuzzy normal congruence on E_s.

□

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